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Conservation laws for divergenceless differential equations

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Abstract. A recent paper by Hojman shows how to use a symmetry to construct a constant of the motion for a system of second-order differential equations directly from the equations, without using either a Lagragian or Hamiltonian. It is, however, necessary to impose some constraint on the nature of these equations. Here this constraint is recognized as saying that the differential equation field is divergenceless with respect to some connection on the extended tangent bundle. However, for the simplest example of a Lagrangian consisting of a kinetic energy and velocity-indepedent potential all the conserved quantities constructed in this way from Noether symmetries are identically zero. The relation of this work to earlier work is pointed out.

In a recent letter Hojman [1] presented a conservation law for a system of second-order differential equations. Under certain assumptions about the equations he obtained a constant of the motion from any symmetry, without using either a Lagrangian or a Hamiltonian. If the equations are written as

$$\ddot{x}^i - F^i(x^j, \dot{x}^j, t) = 0 \qquad i, j = 1, \ldots, n \tag{1}$$

then in the initial statement of his theorem Hojman requires the 'forces' F^i to satisfy

$$\frac{\partial F^i}{\partial \dot{x}^i} = 0 \tag{2}$$

in some coordinates. Later in the paper a generalized statement of the theorem is given in which the forces are required to satisfy the relaxed condition

$$\frac{\partial F^i}{\partial \dot{x}^i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \lambda = 0 \tag{3}$$

where λ is an arbitrary function of the x^i and

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial \dot{x}^i}\,. \tag{4}$$

As Hojman points out, the constraints imposed on the force by (2) or (3) are coordinate-dependent. The class of coordinates in which the constraint is satisfied is singled out as privileged. I will give a geometrical statement of Hojman's conservation

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law in which the assumption of this privileged class of coordinates is replaced by an assumption of the existence of a certain connection on the 'extended' tangent bundle $R \times TM$, where M is the configuration space. Such a geometrical, coordinate-independent statement is of course preferred if we wish to accommodate non-trivial configuration spaces. In addition it will be shown that this geometrical statement of the result is more general than that given by Hojman.

In fact the conservation law presented by Hojman is not new. It had already been presented by Cantrijn and Sarlet [2]. (I am grateful to the referee for drawing my attention to this paper.) I shall conclude by elaborating upon the connection between Hojman's paper, as well as that presented here, and other earlier work.

A system of n second-order autonomous differential equations may be described by a vector field on the tangent bundle TM of an n-dimensional configuration manifold M (see for example [3]). For a system of non-autonomous equations we must generalize to a 'time-dependent vector field', that is a vector field on the 'extended' tangent bundle $R \times TM$. The tangent field to any curve on M is a curve on TM, the natural lift of the curve. The graph of this curve in TM is a curve on $R \times TM$. A second-order non-autonomous differential equation field F is a vector field on $R \times TM$ having the property that its integral curves are the graphs of the natural lifts of their projections to M. If coordinates $\{x^a\}$ $a = 1, \ldots, n$ for M induce natural coordinates $\{x^a, y^a\}$ for TM then F is given by

$$F = \frac{\partial}{\partial t} + F^a \frac{\partial}{\partial y^a} + y^a \frac{\partial}{\partial x^a} \tag{5}$$

where the components F^a are arbitrary functions on $R \times TM$. The projections of integral curves of F from $R \times TM$ to M satisfy the equations (1). A vector field W on $I \times TM$ generates a symmetry of the equations if it commutes with F. In this case F is invariant under the flows of W, and the flows map any integral curve of F to another integral curve, and hence map one solution of the system of equations to another. If X generates a symmetry then in local coordinates we must have

$$X = \alpha \frac{\partial}{\partial t} + X^a \frac{\partial}{\partial x^a} + (FX^a) \frac{\partial}{\partial y^a}$$

where α is a constant of the motion, that is $F\alpha = 0$. Given such a symmetry generator X we can construct another, W, by $W = X - \alpha F$. The symmetry generated by W preserves the fibres of $R \times TM$ above R. In local coordinates

$$W = W^a \frac{\partial}{\partial x^a} + (FW^a) \frac{\partial}{\partial y^a} \tag{6}$$

where the functions W^a satisfy

$$F^2 W^a - FW^b \frac{\partial F^a}{\partial y^b} - W^b \frac{\partial F^a}{\partial x^b} = 0 \tag{7}$$

which is Hojman's equation (4).

Suppose that we had a system of coordinates in which the condition (2) was satisfied. Then we could put a trivial connection on $I \times TM$ by declaring these coordinate vectors to be parallel. Then (2) says that F is divergenceless with respect to this connection. Hojman's conserved quantity I, given by

$$I = \frac{\partial W^a}{\partial x^a} + \frac{\partial}{\partial y^a}(FW^a)$$

is then the divergence of the symmetry generator W with respect to this trivial connection. This suggests that we should replace the coordinate-dependent condition (2) by the requirement that F be divergenceless with respect to some connection on $I \times TM$. The geometrical statement of Hojman's result will follow directly from the following proposition.

Proposition 1. If X and Y are vector fields on a manifold with connection then

$$X\text{div}Y - Y\text{div}X = \text{div}[X,Y] + (\text{div}\,T)(X,Y) - \text{ARic}(X,Y)\,. \qquad (8)$$

The divergence of a vector field Y is defined by $\text{div}Y = e^a\left(\nabla_{X_a} Y\right)$ where the basis $\{e^a\}$ is dual to the arbitrary basis $\{X_a\}$. The torsion tensor of ∇ is T, whose divergence is defined by $(\text{div}T)(X,Y) = \nabla_{X_a} T(X, Y, e^a)$. The Ricci tensor is related to the curvature tensor R of ∇ by $\text{Ric}(X,Y) = R(X_a, X, Y, e^a)$, with the anti-symmetric part defined by $\text{ARic}(X,Y) = \text{Ric}(X,Y) - \text{Ric}(Y,X)$. Although a little tedious, the proof of the proposition is straightforward.

We could use the first Bianchi identity to rewrite the tensor $\text{div}T - \text{ARic}$, but in general this does not lead to anything simple. In the special case of zero torsion we have $\text{ARic}(X,Y) = -R(X,Y,X_a,e^a)$. This will of course be zero for a metric-compatible connection.

Theorem. Let F be a second-order differential equation field that is divergenceless with respect to some connection on $R \times TM$ that satisfies $\text{div}T = \text{ARic}$. Then for any vector field W such that $[W,F] = f_W F$, for some function f_W, a constant of the motion is given by $I_W = \text{div}W + f_W$.

If the torsion and Ricci tensors are related as assumed then Proposition 1 gives

$$W\text{div}F - F\text{div}W = \text{div}[W,F]\,.$$

So if $[W,F] = f_W F$ we have

$$\text{div}[W,F] = \text{div}(f_W F) = f_W \text{div}F + F f_W$$

and

$$W\text{div}F - f_W\text{div}F - F(\text{div}W + f_W) = 0\,.$$

The assumption that $\text{div}F = 0$ then gives $F(\text{div}W + f_W) = 0$, that is $\text{div}W + f_W$ is a constant of the motion.

For the special case of a trivial connection and W generating a symmetry this reduces to the initial statement of Hojman's theorem. Note however that we do not quite require that W generate a symmetry. It is only necessary that the flows of W

scale the differential equation field F by an arbitrary function. The set of vector fields generating these generalized symmetries is a Lie algebra with

$$f_{[W,X]} = W f_X - X f_W \,.$$

We can always add a multiple of F to W to obtain another generalized symmetry $W + hF$ with $f_{W+hF} = f_W - Fh$. In this way we can obtain a true symmetry if we can integrate to find a function h such that $f_W = Fh$. It is easy to see that $I_{W+hF} = I_W$, and so we do not change the constant of the motion in this way.

Initially it might appear that the theorem stated here does not accommodate the generalized statement of Hojman's theorem which can be phrased as follows. If the differential equation field satisfies $\mathrm{div}F = -F \ln \lambda$, where the divergence is with respect to a trivial connection and λ is the lift of some arbitrary function on M, then a constant of the motion is given by $I = (1/\lambda)\mathrm{div}(\lambda W)$ where W generates a symmetry. In fact this assumption about F implies that F is divergenceless with respect to some connection. We will see this from the following.

Proposition 2. If ∇ is a connection on an m-dimensional manifold then for any smooth function f a connection $\hat{\nabla}$ is given by $\hat{\nabla}_X Y = \nabla_X Y + XfY + YfX$. The divergence of an arbitrary vector field Y with respect to this new connection is related to that with respect to the old by $\widehat{\mathrm{div}}Y = \mathrm{div}Y + (m+1)Yf$.

Since $XfY + YfX$ is linear (over the functions) in both X and Y then $\hat{\nabla}$ is indeed a connection. We have

$$\widehat{\mathrm{div}}Y = e^a\left(\hat{\nabla}_{X_a} Y\right) = e^a\left(\nabla_{X_a} Y + X_a f Y + Y f X_a\right) = \mathrm{div}\, Y + Yf + mYf \,.$$

The above proposition 2 lets us recognize Hojman's weaker assumption as saying that F is divergenceless with respect to some connection, related to the trivial one by the function λ. However, to be able to apply the theorem stated here we need to check that the torsion and Ricci tensors of this new connection satisfy the appropriate condition. The following proposition is more general than we actually need for this purpose.

Proposition 3. If a connection $\hat{\nabla}$ is related to another connection ∇ and a smooth function f by $\hat{\nabla}_X Y = \nabla_X Y + XfY + YfX$ then $\hat{T} = T$ and $\widehat{\mathrm{div}}T - \widehat{\mathrm{ARic}} = \mathrm{div}T - \mathrm{ARic}$.

Since $XfY + YfX$ is symmetric in X and Y the torsion tensors of the two connections are the same. Although $\hat{T} = T$ the divergence with respect to $\hat{\nabla}$, $\widehat{\mathrm{div}}T$, will not be the same as $\mathrm{div}T$. In fact we have

$$\begin{aligned}(\widehat{\mathrm{div}}T)(X,Y) = (\mathrm{div}T)(X,Y) + mT(X,Y)f + XfT(Y, X_a, e^a) \\ - YfT(X, X_a, e^a) + T(Y,X)f \,.\end{aligned}$$

The Ricci tensor of $\hat{\nabla}$ is related to that of ∇ by

$$\widehat{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) - (m-1)H_f(Y,X) + (m-1)YfXf + YfT(X_a, X, e^a)$$

where H_f is the Hessian of f (its second covariant derivative). Taking the anti-symmetric part gives

$$\widehat{\mathrm{ARic}}(X,Y) = \mathrm{ARic}(X,Y) + YfT(X_a,X,e^a) - XfT(X_a,Y,e^a) + (m-1)T(X,Y)f\,.$$

So the difference between the anti-symmetrized Ricci tensors is indeed the same as that between the divergences of the torsion tensor.

We can now see how Hojman's weakened statement is accommodated in the theorem stated here. Suppose that the differential equation satisfies $\mathrm{div}F = -F\ln\lambda$ where λ is an *arbitrary* function and the connection satisfies $\mathrm{div}T = \mathrm{ARic}$. Then if $\hat\nabla$ is defined as in proposition 2, with the function f chosen by $f = \ln\lambda/(m+1)$ with m the dimension of the extended tangent bundle, we have $\widehat{\mathrm{div}}F = 0$. Moreover, proposition 3 ensures that $\widehat{\mathrm{div}}T = \widehat{\mathrm{ARic}}$. So from our theorem $\widehat{\mathrm{div}}W$ is a constant of the motion for W any (generalized) symmetry. Using proposition 2 we have $\widehat{\mathrm{div}}W = (1/\lambda)\mathrm{div}(\lambda W)$, and we recover Hojman's expression for the conserved quantity. (Note that there is no need to restrict λ to be a lift of a function on M.)

We have seen that Hojman's constraint implies that the differential equation field is divergenceless; are these conditions just the same? For F as in (5) we have in general

$$\mathrm{div}F = \mathrm{div}\frac{\partial}{\partial t} + \frac{\partial F^a}{\partial y^a} + F^a\mathrm{div}\frac{\partial}{\partial y^a} + y^a\mathrm{div}\frac{\partial}{\partial x^a}$$

and so requiring F to be divergenceless is just Hojman's condition on F if $\mathrm{div}\partial/\partial z = \partial f/\partial z$ for some function f for z any of the coordinate functions, for then $\mathrm{div}F = \partial F^a/\partial y^a + Ff$. This will be the case if the connection is torsion-free and metric-compatible. In that case we have $\mathrm{div}\partial/\partial z = \partial/\partial z \ln\sigma$ where $\sigma = |\det g_{ij}|^{1/2}$. For a general connection the condition that F be divergenceless will not reduce to Hojman's condition. In fact a connection on TM with torsion naturally arises from any connection on M. This connection, called the horizontal lift, is described in [3].

For a (non-degenerate) Lagrangian system all the constants of the motion are related to Cartan symmetries (see for example [3]) and so Hojman's theorem cannot possibly lead to anything new. In fact, as we will show, it is not even very useful in that the constants of the motion associated with Noether symmetries are identically zero. We consider a Lagrangian of the form $L = T - V\circ\pi$ where T is the kinetic energy and $V\circ\pi$ is a potential energy function lifted to TM from a function V on M. In local coordinates $T = \frac{1}{2}g_{ab}y^a y^b$ where g_{ab} are the components of the metric tensor on M in some coordinates. Since we are considering an autonomous system of equations the Euler–Lagrange field will be a vector field on TM, the projection of a 'time-dependent' differential equation field on $R\times TM$. The metric tensor on M lets us construct a metric and corresponding connection on TM. We can use the (pseudo-) Riemannian connection on M to define horizontal lifts to TM. Let $\{H_a\}$ be the horizontal lifts of the basis vectors $\{\partial_a\}$ with $\{V_a\}$ the vertical lifts. In coordinates

$$H_a = \frac{\partial}{x^a} - y^b\Gamma_{ab}{}^c\frac{\partial}{\partial y^c} \qquad V_a = \frac{\partial}{\partial y^a} \tag{9}$$

where $\Gamma_{ab}{}^{c}$ are the connection coefficients of ∇ in the $\{\partial_a\}$ basis. Let the natural dual basis be $\{\theta^a, \phi^a\}$, with $\theta^a(H_b) = \delta^a_b$, $\theta^a(V_b) = 0$ etc. Then a non-degenerate metric on TM is given by

$$G = g_{ab}\theta^a \otimes \theta^b + g_{ab}\phi^a \otimes \phi^b . \tag{10}$$

This metric tensor, and the associated connection, is described by Yano and Ishihara [4], and is sometimes called the Sasaki metric [5]. Let $\boldsymbol{\nabla}$ be the unique torsion-free G-compatible connection. Then one obtains the following:

$$\begin{aligned} &\boldsymbol{\nabla}_{H_a} H_b = \Gamma_{ab}{}^{c} H_c - \tfrac{1}{2} R_{abc}{}^{p} y^c V_p \qquad \boldsymbol{\nabla}_{H_a} V_b = -\tfrac{1}{2} R^q{}_{apb} y^p H_q + \Gamma_{ab}{}^{q} V_q \\ &\boldsymbol{\nabla}_{V_a} H_b = -\tfrac{1}{2} R^q{}_{bpa} y^p H_q \qquad \boldsymbol{\nabla}_{V_a} V_b = 0 \end{aligned} \tag{11}$$

where $R_{abc}{}^{d}$ are the components of the curvature tensor of ∇ in the $\{\partial_a\}$ basis. Note that if M is just R^n with the trivial connection then the Sasaki connection is trivial, with the horizontal and vertical lifts of Cartesian coordinate vectors being parallel.

The Euler–Lagrange field F for the kinetic-plus-potential-energy Lagrangian is simply expressed in terms of vertical and horizontal lifts as

$$F = y^a H_a - g^{ab} \frac{\partial V}{\partial x^b} V_a . \tag{12}$$

It is now straightforward to check that F is divergenceless with the Sasaki connection. So for this important class of differential equations the conditions of the theorem are met. Suppose that K is a Killing vector on M that also leaves the potential V invariant. Then the complete lift $\tilde{K}$ of K to TM generates a (Noether) symmetry. In coordinates

$$\tilde{K} = K^a \frac{\partial}{\partial x^a} + y^b \frac{\partial K^a}{\partial x^b} \frac{\partial}{\partial y^a} . \tag{13}$$

However, if Div is the divergence operator of the Sasaki connection then the connection formulae (11) show that the divergence of a complete lift is related to the divergence on M by

$$\mathrm{Div}\tilde{K} = 2\,\mathrm{div}K \tag{14}$$

where div is the divergence operator on M. If K is a Killing vector field on M then $\mathrm{div}K = 0$ and so the constant of the motion obtained from the theorem is zero.

In the introduction it was noted that Hojman's result was not in fact new. A paper by Cantrijn and Sarlet [2] contains the 'generalized statement' of Hojman's result. Moreover they only require their symmetry generator W to satisfy $[W, F] = f_W F$, rather than commuting with the differential equation F, as required by Hojman. In fact they claim to be extending the work of Lutzky [6], who gave the result for Lagrangian systems and noted that the constant was zero for Noether symmetries. A little later Crampin [7] gave a geometrical statement of the results presented by Cantrijn and Sarlet, noting that the condition imposed on the differential equation could be understood as saying that it was divergenceless. However, his definition of the divergence of a vector field was in terms of the Lie derivative of a globally defined volume-form rather than, as here, in terms of some connection. He noted that for a Lagrangian system a volume-form may be constructed from the Cartan 1-form, and that this volume-form is invariant under the flows of the Euler–Lagrange field. Therefore if the divergence is defined with this volume-form the Euler–Lagrange field will be divergenceless. He went on to show that the conserved quantities of Cantrijn and Sarlet corresponding to Noether symmetries are zero.

The cycle of papers [6, 2, 7] has an interesting precursor. González-Gascón [8] considered a system of first-order differential equations $\dot{x}^i = X^i(x^i)$ and showed how to construct a constant of the motion from any symmetry (in the more general sense of scaling the differential equation field) providing the 'divergence' condition in which $\partial X^i/\partial x^i$ = constant is satisfied. Apparently this result had already been given for the case of a Hamiltonian system [9]. González-Gascón went on [10] to give a geometrical statement of his result, giving a global statement of the 'divergence' condition by defining the divergence in terms of some orienting volume-form. This was further discussed in [11]. Although González-Gascón was considering first-order systems, rather than second-order ones, his method is the same as used by Crampin.

Both González-Gascón and Crampin take the divergence of a vector field to be defined via some orienting volume-form. I have taken the divergence to be defined by a linear connection. The exact relation between these approaches is given by the following.

Proposition 4. If the divergence is defined with a connection ∇ then the following are equivalent:

(i) $X\mathrm{div}Y - Y\mathrm{div}X = \mathrm{div}[X, Y]$

(ii) $\mathrm{div}T = \mathrm{ARic}$

(iii) On each contractible open set U_α there is an n-form ω_α such that $\mathcal{L}_X\omega_\alpha = \mathrm{div}X\omega_\alpha$, where $\mathcal{L}_X$ denotes the Lie derivative. On the intersection $U_\alpha \cap U_\beta$ the n-form ω_α is a constant multiple of ω_β.

Proposition 1 asserts the equivalence of (i) and (ii). Suppose that (iii) is true. Then by Lie-differentiating this relation, using the identity $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$, we arrive at (i). Conversely suppose that (i), and hence (ii), holds. For any point p we may pick some local n-form z defined in the neighbourhood U_α of p. To each vector field X we associate a function $A(X)$ as follows:

$$\mathcal{L}_X z - \mathrm{div}X z = A(X)z\,. \tag{15}$$

The properties of the covariant and Lie derivatives show that the function $A(X)$ depends linearly, over the functions, on X. Therefore the function $A(X)$ is the contraction of a 1-form A on the vector field X. Taking the derivative of (15) with $\mathcal{L}_Y$ produces

$$\mathcal{L}_Y\mathcal{L}_X z - Y\mathrm{div}X z = Y(A(X))z + (\mathrm{div}X + A(X))(\mathrm{div}Y + A(Y))z\,.$$

If we antisymmetrize in X and Y, noting that the commutator of the Lie derivatives is the derivative of the commutator, we get

$$\mathrm{div}[Y, X] - (Y\mathrm{div}X - X\mathrm{div}Y) = YA(X) - XA(Y) - A([Y, X]) = 2dA(Y, X)\,.$$

So if (i) holds the 1-form A is closed. By the Poincaré lemma it is therefore exact on any contractible neighbourhood. If on U_α we have $A = \mathrm{d}f_\alpha$ then

$$\mathcal{L}_X z = \mathrm{div}X\, z + X f_\alpha z$$

and so

$$\mathcal{L}_X (\exp(-f_\alpha) z) = \mathrm{div}X (\exp(-f_\alpha) z) .$$

Thus (i) implies (iii).

Notice that even on an orientable manifold (i) does not imply that there is a global volume-form ω such that $\mathcal{L}_X \omega = \mathrm{div}X\omega$. If the 1-form A in (15) is not exact then no such global n-form can exist. As a simple example consider the punctured plane. Here the 1-form A given in local polar coordinates (r, θ) by $A = \mathrm{d}\theta$ is closed but not exact. If ∇ is the standard R^2 connection then we may introduce another connection $\hat{\nabla}$ by

$$\hat{\nabla}_X Y = \nabla_X Y + A(Y) X .$$

The divergence operators are related by $\hat{\mathrm{div}}X = \mathrm{div}X + 2A(X)$. If ω is the standard volume-form then $\mathcal{L}_X \omega = \mathrm{div}X\omega$ and so $\mathcal{L}_X \omega = (\hat{\mathrm{div}}X - 2A(X))\omega$. On any contractible region U_α we may multiply ω by the exponential of a local 'angle' θ_α to form a local form $\omega_\alpha = \exp(2\theta_\alpha)\omega$ such that $\mathcal{L}_X \omega_\alpha = \hat{\mathrm{div}}X \omega_\alpha$. As we go around the circle these local forms differ by a constant multiple on overlaps. Although this manifold is orientable there is no global volume-form z such that $\mathcal{L}_X z = \hat{\mathrm{div}}X z$, but we do have $X\hat{\mathrm{div}}Y - Y\hat{\mathrm{div}}X = \hat{\mathrm{div}}[X, Y]$.

The unquestionable importance of constants of the motion for systems of differential equations means that any prescription for constructing such constants is worthy of careful scrutiny. In this paper we have given a coordinate-invariant statement of the theorem recently presented by Hojman. Such a statement is not only to be preferred to accommodate non-trivial configuration spaces but is also more general. For a torsion-free metric-compatible connection the divergenceless condition is exactly Hojman's 'weaker' constraint. Also we only need a 'generalized' symmetry that scales the differential equation field rather than leaving it invariant. It transpires that Hojman's results are not in fact new, having already been given by Cantrijn and Sarlet. In turn their results could perhaps have been anticipated from the work on first-order systems by González-Gastón. Many of the points that we have made here in relation to Hojman's paper were made by Crampin in relation to that of Cantrijn and Sarlet. Unlike Crampin our definition of the divergence operator is based on some connection rather than a global volume-form. This allows a slightly more general statement of the result. However, certainly for the case of Lagrangian systems the Cartan 1-form does lead naturally to a divergence operator defined directly in terms of a volume-form.

To use the result it is not only necessary to find a symmetry generator but also to find some connection leaving the differential equation field divergenceless. Unfortunately in the simplest of cases this result is useless in that for Noether symmetries the 'new' constant is zero. That is not to say that there could not be other symmetries leading to non-zero constants (Hojman gives an example) but in general finding symmetry generators is akin to solving the original equations! Another possibility might be that there are other less obvious connections with respect to which the differential equations are divergenceless and for which the Noether symmetries lead to non-zero constants.

It remains to be seen if, say for non-Lagrangian systems, this result will actually be useful in finding new constants of the motion.

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